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Synchrotron Production of Photons by a Two-Body System

Miroslav Pardy[1]

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The power spectrum of the synchrotron radiation generated by the motion of a two-body charged system in an accelerator is derived in the framework of the Schwinger source theory. The final formula can be used to verify the Lorentz length contraction of the two-body system moving in the synchrotron.

The production of photons by the circular motion of charged particles in an accelerator is one of the most interesting problems in the classical and quantum electrodynamics.

In this paper we are interested in the photon production initiated by circular motion and a two-body charged system. This process specifies the synergic synchrotron Čerenkov radiation, which was calculated in source theory two decades ago by Schwinger *et al.* (1976). We will follow also Pardy (1994a). The synergic process includes the effect of the medium, which is represented by the phenomenological index of refraction n, and it is well known that this phenomenological constant depends on the external magnetic field.

We will investigate, first, how the original Schwinger *et al.* spectral formula of the synergic synchrotron Čerenkov radiation of a charged particle moving in a medium is modified if we consider a two-body system. Then, we will treat this process in vacuum. This problem is an analogue of the linear problem solved recently by the author (Pardy, 1997) also in source theory. We will show that the original spectral formula of the synergic synchrotron-Čerenkov radiation is modulated by the function $\cos^2(a\omega/2v)$, where a

[1] Department of Theoretical Physics and Astrophysics, Faculty of Science, Masaryk University, 611 37 Brno, Czech Republic; e-mail: pamir@physics.muni.cz.

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is the distance between charges, v is their velocity, and ω is the frequency of the synergic radiation produced by the system.

Source theory (Schwinger, 1970, 1973; Dittrich, 1978) was initially constructed for a description of high-energy particle physics experiments. It was found that the original formulation simplifies calculations in electrodynamics and gravity, where the interactions are mediated by the photon and graviton, respectively. It simplifies particularly calculations with radiative corrections (Dittrich, 1978; Pardy, 1994b).

The basic formula of the Schwinger source theory is the so-called vacuum to vacuum amplitude:

$$\langle 0_+|0_-\rangle = e^{(i/\hbar)W} \tag{1}$$

where, for the case of an electromagnetic field in the medium, the action W is given by

$$W = \frac{1}{2c^2}\int (dx)(dx')J^{\mu}(x)D_{+\mu\nu}\,(x - x')J^{\nu}(x') \tag{2}$$

where

$$D_+^{\mu\nu} = \frac{\mu}{c}\,[g^{\mu\nu} + (1 - n^{-2})\beta^{\mu}\beta^{\nu}]\,D_+(x - x') \tag{3}$$

and $\beta^{\mu} \equiv (1, \mathbf{0})$, $J^{\mu} \equiv (c\rho, \mathbf{J})$ is the conserved current, μ is the magnetic permeability of the medium, $\in$ is the dielectric constant of the medium, and $n = \sqrt{\in\mu}$ is the index of refraction of the medium. The function D_+ is defined as follows (Schwinger *et al.*, 1976):

$$D_+(x - x') = \frac{i}{4\pi^2 c}\int_0^{\infty} d\omega\,\frac{\sin(n\omega/c)|\mathbf{x} - \mathbf{x}'|}{|\mathbf{x} - \mathbf{x}'|}\,e^{-i\omega|t-t'|} \tag{4}$$

The probability of the persistence of the vacuum follows from the vacuum amplitude (1) in the following form:

$$|\langle 0_+|0_-\rangle|^2 = e^{-(2/\hbar)\mathrm{Im}W} \tag{5}$$

where Im W is the basis for the definition of the spectral function $P(\omega, t)$ as follows:

$$-\frac{2}{\hbar}\,\mathrm{Im}\,W \stackrel{d}{=} -\int dt\,d\omega\,\frac{P(\omega, t)}{\hbar\omega} \tag{6}$$

Now, if we insert Eq. (2) into Eq. (6), we get, after extracting $P(\omega, t)$, the following general expression for this spectral function:

$$P(\omega, t) = -\frac{\omega}{4\pi^2}\frac{\mu}{n^2}\int d\mathbf{x}\, d\mathbf{x}'\, dt' \left[\frac{\sin(n\omega/c)\,|\mathbf{x}-\mathbf{x}'|}{|\mathbf{x}-\mathbf{x}'|}\right]$$
$$\times \cos[\omega(t-t')]\,[\rho(\mathbf{x}, t)\rho(\mathbf{x}', t') - \frac{n^2}{c^2}\mathbf{J}(\mathbf{x}, t)\cdot \mathbf{J}(\mathbf{x}', t')] \quad (7)$$

Now, we will apply the formula (7) to the two-body system with the same charged particles in order to get its synergic synchrotron-Čerenkov radiation. The synchrotron radiation is produced by a particle of charge e moving in a uniform circular motion with velocity $\mathbf{v}$ in the plane perpendicular to the direction of the constant magnetic field $\mathbf{H}$ (chosen to be in the $+z$ direction).

On the other hand, the Čerenkov electromagnetic radiation is generated by a fast-moving charged particle in a medium when its speed is faster than the speed of light in this medium. This radiation was first observed experimentally by Čerenkov (1936) and theoretically interpreted by Tamm and Frank (1937) in the framework of classical electrodynamics. A source-theoretic description of this effect was given by Schwinger *et al.* (1976) in the zero-temperature regime, and the classical spectral formula was generalized to the finite-temperature situation in electrodynamics and gravity in the framework of the source theory by Pardy (1989, 1995).

In electrodynamics one usually considers synchrotron radiation produced by a uniformly moving charge with constant orbital velocity. Here we consider the system of two equal charges e with constant mutual distance a moving with orbital velocity v in the accelerator. For the sake of generality we consider also that a dielectric medium is present. So we write for the charge density ρ and for the current density $\mathbf{J}$ of the two-body system

$$\rho(\mathbf{x}, t) = e\delta(\mathbf{x} - \mathbf{x}_1(t)) + e\delta(\mathbf{x} - \mathbf{x}_2(t)) \quad (8)$$

and

$$\mathbf{J}(\mathbf{x}, t) = e\mathbf{v}_1(t)\delta(\mathbf{x} - \mathbf{x}_1(t)) + e\mathbf{v}_2(t)\delta\,(\mathbf{x} - \mathbf{x}_2(t)) \quad (9)$$

with

$$\mathbf{x}_1(t) = \mathbf{x}(t) = R(\mathbf{i}\cos(\omega_0 t) + \mathbf{j}\sin(\omega_0 t)) \quad (10)$$

$$\mathbf{x}_2(t) = R(\mathbf{i}\cos(\omega_0 t + \delta\varphi) + \mathbf{j}\sin(\omega_0 t + \delta\varphi)) = \mathbf{x}\left(t + \frac{\delta\varphi}{\omega_0}\right);$$

$$\delta\varphi = \frac{a}{R} \quad (11)$$

We will suppose for simplicity that the distance between the particles forming the two-body system is very small in comparison with the diameter R of the circular accelerator, which means that the velocities of both particles are approximately the same, or $\mathbf{v}_1(t) \approx \mathbf{v}_2(t) = \mathbf{v}(t)$, where ($H = |\mathbf{H}|$, $E =$ energy of a particle)

$$\mathbf{v}(t) = d\mathbf{x}/dt, \qquad \omega_0 = v/R, \qquad R = \beta E/eH, \qquad \beta = v/c, \qquad v = |\mathbf{v}| \tag{12}$$

After insertion of Eqs. (8) and (9) into Eq. (7), and after some mathematical operations, we get

$$\begin{aligned} P(\omega, t) = {} & -\frac{\omega}{4\pi^2}\frac{\mu}{n^2} e^2 \int_{-\infty}^{\infty} dt' \cos(t - t') \left[1 - \frac{\mathbf{v}(t)\cdot\mathbf{v}(t')}{c^2} n^2\right] \\ & \times \left\{ \frac{\sin(n\omega/c)|\mathbf{x}_1(t) - \mathbf{x}_1(t')|}{|\mathbf{x}_1(t) - \mathbf{x}_1(t')|} + \frac{\sin(n\omega/c)|\mathbf{x}_1(t) - \mathbf{x}_2(t')|}{|\mathbf{x}_1(t) - \mathbf{x}_2(t')|} \right. \\ & \left. + \frac{\sin(n\omega/c)|\mathbf{x}_2(t) - \mathbf{x}_1(t')|}{|\mathbf{x}_2(t) - \mathbf{x}_1(t')|} + \frac{\sin(n\omega/c)|\mathbf{x}_2(t) - \mathbf{x}_2(t')|}{|\mathbf{x}_2(t) - \mathbf{x}_2(t')|} \right\} \end{aligned} \tag{13}$$

Using $t' = t + \tau$, we get

$$\mathbf{x}_1(t) - \mathbf{x}_1(t') = \mathbf{x}(t) - \mathbf{x}(t + \tau) \stackrel{d}{=} \mathbf{A} \tag{14}$$

$$\mathbf{x}_1(t) - \mathbf{x}_2(t') = \mathbf{x}(t) - \mathbf{x}\left(t + \tau + \frac{\delta\varphi}{\omega_0}\right) \stackrel{d}{=} \mathbf{B} \tag{15}$$

$$\mathbf{x}_2(t) - \mathbf{x}_1(t') = \mathbf{x}\left(t + \frac{\delta\varphi}{\omega_0}\right) - \mathbf{x}(t + \tau) \stackrel{d}{=} \mathbf{C} \tag{16}$$

$$\mathbf{x}_2(t) - \mathbf{x}_2(t') = \mathbf{x}\left(t + \frac{\delta\varphi}{\omega_0}\right) - \mathbf{x}\left(t + \tau + \frac{\delta\varphi}{\omega_0}\right) \stackrel{d}{=} \mathbf{D} \tag{17}$$

Using the geometrical representation of vectors $\mathbf{x}_i(t)$, we get

$$|\mathbf{A}| = [R^2 + R^2 - 2RR\cos(\omega_0\tau)]^{1/2} = 2R\left|\sin\left(\frac{\omega_0\tau}{2}\right)\right| \tag{18}$$

$$|\mathbf{B}| = 2R\left|\sin\left(\frac{\omega_0\tau + \delta\varphi}{2}\right)\right| \tag{19}$$

$$|\mathbf{C}| = 2R\left|\sin\left(\frac{\omega_0\tau - \delta\varphi}{2}\right)\right| \tag{20}$$

$$|\mathbf{D}| = 2R\left|\sin\left(\frac{\omega_0\tau}{2}\right)\right| \tag{21}$$

Using

$$\mathbf{v}(t \cdot \mathbf{v}(t + \tau) = \omega_0^2 R^2 \cos \omega_0 \tau \tag{22}$$

and relations (18)–(21), we get, with $v = \omega_0 R$,

$$\begin{aligned} P(\omega, t) &= -\frac{\omega}{4\pi^2}\frac{\mu}{n^2}e^2 \int_{-\infty}^{\infty} d\tau \cos\omega\tau \left[1 - \frac{n^2}{c^2} v^2 \cos\omega_0\tau\right] \\ &\times \left\{ \frac{\sin[(2Rn\omega/c)\sin(\omega_0\tau/2)]}{2R\sin(\omega_0\tau/2)} + \frac{\sin[(2Rn\omega/c)\sin((\omega_0\tau + \delta\varphi)/2)]}{2R\sin((\omega_0\tau + \delta\varphi)/2)} \right. \\ &\left. + \frac{\sin[(2Rn\omega/c)\sin((\omega_0\tau - \delta\varphi)/2)]}{2R\sin((\omega_0\tau - \delta\varphi)/2)} + \frac{\sin[(2Rn\omega/c)\sin(\omega_0\tau/2)]}{2R\sin(\omega_0\tau/2)} \right\} \end{aligned} \tag{23}$$

Introducing the new variable T by the relation

$$\omega_0\tau + \alpha_i = \omega_0 T \tag{24}$$

for every integral in Eq. (23), where

$$\alpha_i = 0, \delta\varphi, -\delta\varphi, 0 \tag{25}$$

we get $P(\omega, t)$ in the following form:

$$\begin{aligned} P(\omega, t) = -\frac{\omega}{4\pi^2}\frac{e^2}{2R}\frac{\mu}{n^2}\int_{-\infty}^{\infty} dT \\ \times \sum_{i=1}^{4} \cos\left(\omega T - \frac{\omega}{\omega_0}\alpha_i\right)\left[1 - \frac{c^2}{n^2} v^2 \cos(\omega_0 T - \alpha_i)\right] \\ \times \left\{\frac{\sin[2Rn\omega/c)\ \sin(\omega_0 T/2)]}{\sin(\omega_0 T/2)}\right\} \end{aligned} \tag{26}$$

The last formula can be written in the more compact form

$$P(\omega, t) = -\frac{\omega}{4\pi^2}\frac{\mu}{n^2}\frac{e^2}{2R}\sum_{i=1}^{4}\left\{P_1^{(i)} - \frac{n^2}{c^2} v^2 P_2^{(i)}\right\} \tag{27}$$

where

$$P_1^{(i)} = J_{1a}^{(i)} \cos\frac{\omega}{\omega_0}\,\alpha_i + J_{1b}^{(i)} \sin\frac{\omega}{\omega_0}\,\alpha_i \tag{28}$$

and

$$\begin{aligned} P_2^{(i)} = {} & J_{2A}^{(i)} \cos\alpha_i \cos\frac{\omega}{\omega_0}\,\alpha_i \\ & + J_{2B}^{(i)} \cos\alpha_i \sin\frac{\omega}{\omega_0}\,\alpha_i + J_{2C}^{(i)} \sin\alpha_i \cos\frac{\omega}{\omega_0}\,\alpha_i \\ & + J_{2D}^{(i)} \sin\alpha_i \sin\frac{\omega}{\omega_0}\,\alpha_i \end{aligned} \tag{29}$$

where

$$J_{1a}^{(i)} = \int_{-\infty}^{\infty} dT \cos\omega T \left\{\frac{\sin[2Rn\omega/c)\,\sin(\omega_0 T/2)]}{\sin(\omega_0 T/2)}\right\} \tag{30}$$

$$J_{1b}^{(i)} = \int_{-\infty}^{\infty} dT \sin\omega T \left\{\frac{\sin[2Rn\omega/c)\,\sin(\omega_0 T/2)]}{\sin(\omega_0 T/2)}\right\} \tag{31}$$

$$J_{2A}^{(i)} = \int_{-\infty}^{\infty} dT \cos\omega_0 T \cos\omega T \left\{\frac{\sin[(2Rn\omega/c)\,\sin(\omega_0 T/2)]}{\sin(\omega_0 T/2)}\right\} \tag{32}$$

$$J_{2B}^{(i)} = \int_{-\infty}^{\infty} dT \cos\omega_0 T \sin\omega T \left\{\frac{\sin[(2Rn\omega/c)\,\sin(\omega_0 T/2)]}{\sin(\omega_0 T/2)}\right\} \tag{33}$$

$$J_{2C}^{(i)} = \int_{-\infty}^{\infty} dT \sin\omega_0 T \cos\omega T \left\{\frac{\sin[(2Rn\omega/c)\,\sin(\omega_0 T/2)]}{\sin(\omega_0 T/2)}\right\} \tag{34}$$

$$J_{2D}^{(i)} = \int_{-\infty}^{\infty} dT \sin\omega_0 T \sin\omega T \left\{\frac{\sin[(2Rn\omega/c)\,\sin(\omega_0 T/2)]}{\sin(\omega_0 T/2)}\right\} \tag{35}$$

Using

$$\omega_0 T = \varphi + 2\pi l, \qquad \varphi \in (-\pi, \pi), \quad l = 0, \pm 1, \pm 2, \ldots \tag{36}$$

we can transform the T-integral into the sum of the telescopic integrals according to the scheme

$$\int_{-\infty}^{\infty} dT \rightarrow \frac{1}{\omega_0} \sum_{-\infty}^{\infty} \int_{-\pi}^{\pi} d\varphi \tag{37}$$

Using the fact that for the odd functions $f(\varphi)$ and $g(l)$, the following relations are valid,

$$\int_{-\pi}^{\pi} f(\varphi)\, d\varphi = 0; \qquad \sum_{l-\infty}^{\infty} g(l) = 0 \tag{38}$$

we can write

$$J_{1a}^{(i)} = \frac{1}{\omega_0} \sum_l \int_{-\pi}^{\pi} d\varphi \left\{ \cos \frac{\omega}{\omega_0} \varphi \cos 2\pi l \frac{\omega}{\omega_0} \right\} \left\{ \frac{\sin[(2Rn\omega/c) \sin(\varphi/2)]}{\sin(\varphi/2)} \right\} \tag{39}$$

$$J_{1b}^{(i)} = 0 \tag{40}$$

For integrals with indices A, B, C, D we get

$$J_{2A}^{(i)} = \frac{1}{\omega_0} \sum_l \int_{-\pi}^{\pi} d\varphi \cos \varphi \left\{ \cos \frac{\omega}{\omega_0} \varphi \cos 2\pi l \frac{\omega}{\omega_0} \right\} \times \left\{ \frac{\sin[(2Rn\omega/c) \sin (\varphi/2)]}{\sin(\varphi/2)} \right\} \tag{41}$$

$$J_{2B}^{(i)} = J_{2C}^{(i)} = 0 \tag{42}$$

$$J_{2D}^{(i)} = \frac{1}{\omega_0} \sum_l \int_{-\pi}^{\pi} d\varphi \sin \varphi \left\{ \sin \frac{\omega}{\omega_0} \varphi \cos 2\pi l \frac{\omega}{\omega_0} \right\} \times \left\{ \frac{\sin[(2Rn\omega/c) \sin(\varphi/2)]}{\sin(\varphi/2)} \right\} \tag{43}$$

Using the Poisson theorem,

$$\sum_{k=-\infty}^{\infty} \cos 2\pi \frac{\omega}{\omega_0} k = \sum_{k=-\infty}^{\infty} \omega_0 \delta(\omega - \omega_0 k) \tag{44}$$

the definition of the Bessel functions J_{2l}, and their corresponding derivation and integral

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi \cos \left(z \sin \frac{\varphi}{2} \right) \cos l\varphi = J_{2l}(z) \tag{45}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi \sin \left(z \sin \frac{\varphi}{2} \right) \cos l\varphi = -J'_{2l}(z) \tag{46}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi\, \frac{\sin(z \sin(\varphi/2))}{\sin(\varphi/2)} \cos l\varphi = \int_0^z J_{2l}(x)\, dx \tag{47}$$

we get, with ($a \ll R$)

$$\sin \alpha_i \sin \frac{\omega}{\omega_0} \alpha_i \approx 0 \tag{48}$$

and with the definition of the partial power spectrum P_l,

$$P(\omega) = \sum_{l=1}^{\infty} \delta(\omega - l\omega_0) P_l \tag{49}$$

the following final form of the partial power spectrum generated by motion of a two-charge system moving in the cyclotron:

$$P_l(\omega, t) = \cos^2\left(\frac{a\omega}{2v}\right) \frac{e^2}{\pi n^2} \frac{\omega\mu\omega_0}{v} \left(2n^2\beta^2 J'_{2l}\,(2ln\beta) - (1 - n^2\beta^2) \int_0^{2ln\beta} dx\, J_{2l}(x)\right) \tag{50}$$

Our goal is to apply the last formula to the situation when the medium of the accelerator is in fact a vacuum. In this case we can put $\mu = 1$ and $n = 1$ in the last formula and so we have

$$P_l = \cos^2\left(\frac{a\omega}{2v}\right) \frac{e^2}{\pi} \frac{\omega\omega_0}{v} \left(2\beta^2 J'_{2l}(2l\beta) - (1 - \beta^2) \int_0^{2l\beta} dx\, J_{2l}(x)\right) \tag{51}$$

Using the approximative formulas

$$J'_{2l}\,(2l\beta) \sim \frac{1}{\sqrt{3}} \frac{1}{\pi} \left(\frac{3}{2l_c}\right)^{2/3} K_{2/3}(l/l_c), \qquad l \gg 1 \tag{52}$$

$$\int_0^{2l\beta} J_{2l}(y)\, dy \sim \frac{1}{\sqrt{3}} \frac{1}{\pi} \int_{l/l_c}^{\infty} K_{1/3}(y)\, dy, \qquad l \gg 1 \tag{53}$$

with (Schwinger *et al.*, 1976)

$$l_c = \frac{3}{2}(1 - \beta^2)^{-3/2} \tag{54}$$

substituting Eqs. (52) and (53) into Eq. (51), respecting the high-energy situation for the high-energy particles where $(1 - \beta^2) \to 0$, and using the recurrence relation

$$K'_{2/3} = -\frac{1}{2}(K_{1/3} + K_{5/3}) \tag{55}$$

and definition of the function $\kappa(\xi)$

$$\kappa(\xi) = \xi \int_{\xi}^{\infty} K_{5/3}(y)\, dy, \qquad \xi = l/l_c \tag{56}$$

we get the power spectrum of an electron–electron pair as follows:

$$P(\omega) = \cos^2\left(\frac{a\omega}{2v}\right) \frac{\omega e^2}{\pi^2 R} \sqrt{\frac{\pi}{6}} \left(\frac{3}{2l}\right)^{2/3} \xi^{1/6}\, e^{-\xi}; \qquad l = \frac{\omega}{\omega_0} \tag{57}$$

where we used the idea that the discrete spectrum parametrized by the number l is effectively continuous for $l \gg 1$. In such a case we have

$$P(\omega) = P_{(l=\omega/\omega 0)} \left(\frac{1}{\omega_0}\right) \tag{58}$$

Formula (57) is analogous to the formula derived in Pardy (1997) for the linear motion of a two-charge system emitting Čerenkov radiation.

The radiative corrections obviously influence the spectrum (Schwinger, 1970; and Pardy, 1994b). Determination of this phenomenon forms a special problem of accelerator physics.

Use of large accelerators, for instance, Grenoble, DESY, or CERN, should make possible experimental verification of the derived formulas involving also the Lorentz contraction. Instead of two electrons we can consider, say, two bunches with 10^{10} electrons in each bunch of volume 300 μm $\times$ 40 μm $\times$ 0.01 m, with a rest distance $l = 1$ m between them. The distance between the two bunches is the relativistic length a and it can be determined by the synchrotron spectrum derived here.

The results of the Lorentz contraction measurement obtained from the synchrotron radiation spectrum in vacuum ($n = 1$, $\mu = 1$) should be identical with the results of a measurement obtained from a spectrum generated by linear uniform particle motion in a medium because the interference of light in vacuum does not differ from the interference of light emitted by the Čerenkov effect in a medium.

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